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# APPLICATIONS OF RANDOM SET REPRESENTATIONS OF FUZZY SETS TO DETERMINING MEASURES OF CENTRAL TENDENCY


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# Applications Of Random Set Representations Of Fuzzy Sets To Determining Measures Of Central Tendency

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## Abstract
The purpose of this paper is to both summarize the progress made so far in the problem of determining measures of central tendency of fuzzy sets and to propose a new approach based on the author's recent work on characterizing homomorphic-like operators among fuzzy sets and related random sets [0].

## 1. Introduction

Measures of central tendency as used in this abstract refer to the domain values of a given fuzzy set, not to the range values, such as considered by Schneider [1]. Two motivating reasons for determining measures of central tendency of fuzzy sets arise from the need to rank or compare fuzzy sets [2] and to be able to "defuzzify"; i.e., make crisp, output fuzzy sets in fuzzy control [3],[4]. In addition, other problems often arise which require the presence of a single figure-of-merit value which "best" represents a fuzzy set model output. An example of this is to be found in the author's work on the PACT (Possibilistic Approach to Correlation and Tracking) algorithm, the basic outputs of which (before further processing) are fuzzy subsets of the unit interval representing the "correlation" or degree of association among any given pair of target track histories which can potentially represent the same target. (See, [5], pp. 104-112 and [6].)

## 2. MOM and COA Approaches

In the works cited above, many of the proposed measures of central tendency are developed in an ad hoc manner. Among these, one basic approach is the MOM (mean-of-maxima of the fuzzy set membership function) and another is the COA (center-of-area) approach, also originally proposed by Yager [7]. These, are developed as formal analogies to the standard measures of central tendency in probability theory, such as the mean, median, or mode. This is because there is a well-established fuzzy "folklore" tradition of re-normalizing fuzzy sets via their sums of their membership functions (in the discrete case) to become probability

functions, or renormalizing via the areas under their curves above the zero axis (in the continuous case), to become probability density functions. However, a main drawback in this approach is the obvious one: If $f:D \rightarrow [0,1]$ is any given fuzzy set membership function and $0 < t < 1$ is any real number, both f and the product of t with f -- $t \cdot f$ -- will have the same MOM and COA measures. Yet, as we let t approach 0, intuitively, the product form tf should have a zero or other extremal measure, not the same as the nontrivial one for f.

Both the MOM and COA approaches have been generalized by Yager and Filev's BADD (basic defuzzification distribution) approach [4], whereby renormalization is applied to powers of the fuzzy set membership function in question. (See also the earlier discussions on the use of these approaches in Tong [8] and Dubois & Prade [9], p. 302.) However, fuzzy sets are not the same, in general, as probability functions or probability density functions, although they are intimately related to probability through the distinct concept of random sets.

## 3. Use of One-Point Coverage Relations Between Fuzzy Sets and Random Sets

A growing number of researchers in fuzzy set theory (see, e.g., Dubois & Prade [10] and Zwick [11] have begun to recognize the fundamental relations between fuzzy (sub)sets of a set D, $f:D \rightarrow [0,1]$, and corresponding (highly nonunique in general) random (sub)sets of D $S_f: \Omega \rightarrow P(D)$, relative to some fixed probability space $(\Omega, A, P)$,which are *one-point coverage-equivalent to f*, i.e.,

$$P(x \text{ in } S_f) = f(x) \text{ , all } x \text{ in } D \text{ .} \tag{1}$$

Obviously, any random set generates a uniquely corresponding one-point coverage-equivalent fuzzy subset via eq.(1). Conversely, for any given f, eq.(1) is always satisfied by, in general, many $S_f$. In particular, recalling that the *level sets*, or *alpha-cuts*, of f are of the nested (crisp) set form

$$f^{-1}[t,1] = \{x \text{ in } D: t \leq f(x) \leq 1\} \quad , 0 \leq t \leq 1 \text{ ,} \tag{2}$$

define the *uniformly random level set*

$$S_{f,U} = f^{-1}[U,1] : \Omega \rightarrow P(D) \text{ ,} \tag{3}$$

where $U:\Omega \rightarrow [0,1]$ is a uniformly distributed random variable. Then, the uniformly random level set is clearly one-point coverage-equivalent to f. (This concept began with early work of Höhle, Goodman, Orlov and others; see, e.g. [12] and [13], Chapter 5 for history and background of this area.)

We illustrate further the above idea concerning the identification of a fuzzy set through all one-point coverage-equivalent random subsets of the function's domain -- or equivalently, as a *weak specification of a random set* via its local

behavior (the one-point coverage function): Another quite distinct example of a random set $S_f$ one-point coverage-equivalent to given fuzzy set f, which has a finite domain D is furnished by first constructing an independent zero-one stochastic process $Y_f = (Y_{f,x}: x \text{ in } D\}$, where

$$P(Y_{f,x} = 1) = f(x) \quad \text{and} \quad P(Y_{f,x} = 0) = 1\text{-}f(x)\ , \text{ all x in D.} \qquad (3')$$

It is readily seen that $Y_f$ corresponds to a random subset $S_f$ of D, so that denoting the ordinary set membership or indicator functional as $\phi$, (i.e., $\phi(S_f) = Y_f$, so that:

$$\text{x is covered by } S_f \quad \text{iff x in } S_f \quad \text{iff } \phi(S_f)(x) = 1 \text{ iff } Y_{f,x} = 1$$

and (3")

$$\text{x is not covered by } S_f \text{ iff x not in } S_f \text{ iff } \phi(S_f)(x) = 0 \text{ iff } Y_{f,x} = 0\ .$$

In turn, (3") shows that for each x in D, independently,

$$P(\text{x is covered by } S_f) = P(\text{x in } S_f) = P(\phi(S_f)(x) = 1) = P(Y_{f,x} = 1) = f(x), \quad (3''')$$

showing, in turn, the matching of the one-point coverage function of this $S_f$ with f itself. Because of the joint independence assumption, $S_f$ is a very broken-up random set, unlike the nested form of $S_{f,U}$ above. However, both $S_{f,U}$ and $S_f$ are one-point coverage-equivalent to f. In fact, there are many other such random subsets of D which can be constructed to match f via their one-point coverage functions by choosing, in effect, appropriate joint behavior of the $Y_{f,x}$, replacing the independence assumption (See Section 4 of this paper for related concepts and Goodman [0] for a full solution to this problem.)

Despite the appealing random set connection with fuzzy sets outlined above, applications of this relationship to fuzzy set issues have been relatively sparse, including the issue of determining measures of central tendency for fuzzy sets. Among those who have considered the use of one-point coverage -equivalent random sets, mention must be made of the work of Goodman [5], Dubois & Prade [14], Chanas & Nowakowski [15], Kaufmann & Gupta [16], González [17], and Heilpern [18] (which will all be briefly discussed later). However, these individuals either restricted their attention only to the uniformly random level set (as in [5], where the expected centroid of $S_{f,U}$ was sought) or to fuzzy numbers, or equivalently, fuzzy intervals - i.e., fuzzy sets which essentially have all of their level sets being closed bounded intervals - as in the case of [14]-[18], where emphasis was given to the "mean" of a fuzzy set interpreted through the mean of corresponding one-point coverage equivalent random sets.

Specifically, Kaufmann & Gupta [16], and independently, Chanas & Nowakowski developed the same idea, as, in effect, particular evaluations of the result of Dubois & Prade [14], the latter based on the use of the uniformly random level set representation of a fuzzy set, Dempster's upper and lower probabilities [19], and Artstein & Vitale's definition of the expectation of a random set [20]:

Let $(\Omega, \mathcal{A}, P)$ be a fixed probability space and $S: \Omega \to \mathcal{P}(D)$ any random subset of D. Call any (measurable) $g: \Omega \to D$ a *selection* of S iff (up to P-probability 1)

$$g(\omega) \text{ in } S(\omega), \text{ all } \omega \text{ in } \Omega, \tag{4}$$

and consider the induced probability measure $P \circ g^{-1}: \mathcal{B} \to [0,1]$, by g with respect to P, where $\mathcal{B} \subseteq \mathcal{P}(D)$ and, assuming $D \subseteq \mathbb{R}$, the corresponding probability distribution function $F_{P,g}: \mathbb{R} \to [0,1]$, where denoting the infinite closed left ray at x as $(-\infty, x]$, for any x in $\mathbb{R}$,

$$F_{P,g}(x) = (\text{def})\ P(g^{-1}(-\infty, x]) \tag{5}$$

with corresponding expectation

$$E(P \circ g^{-1}) = \int_{x=-\infty}^{+\infty} x \, dF_{P,g}(x) = \int_{\omega \text{ in } D} g(\omega) dP(\omega) . \tag{6}$$

Define $\mathcal{Q}_S$ as the class of all induced probability measures by g with respect to P, for all possible selections g of S with the notation that for any A in $\mathcal{B}$,

$$\mathcal{Q}_S(A) = (\text{def})\ \{P(g^{-1}(A)) : g \text{ a selection of } S\}. \tag{7}$$

Further, define (Artstein and Vitale) E(S), the *expectation* of S as the class of all expectations of such induced probability measures. It should be remarked that at least for the case of $D \subseteq \mathbb{R}$, D finite, this definition is readily seen to coincide with the functional image extension definition:

$$E(S) = (\text{def}) \sum_{A \subseteq D} A \cdot P(S=A) = (\text{def}) \left\{ \sum_{A \subseteq D} x_A \cdot P(s=A) : x_A \text{ in } A \subseteq D \right\} \tag{8}$$

Recall the *belief and plausibility functions* corresponding to S (see, e.g. Shafer [21]) $bel_S, plaus_S: \mathcal{B} \to [0,1]$, where

$$bel_S(A) = (\text{def})\ P(\emptyset \neq S \subseteq A) \leq P(S \cap A \neq \emptyset) = (\text{def})\ plaus_S(A), \tag{9}$$

for any A in $\mathcal{B}$. Also, note that the *upper and lower probability distribution functions* of S, respectively, are $F^*_S, F_{*S}: \mathbb{R} \to [0,1]$ which are legitimate probability distribution functions, where, for all x in $\mathbb{R}$ (real line),

$$F^*_S(x) =(\text{def})\ \text{plaus}_S((-\infty,x])\ ,\ \ F_{*S}(x) =(\text{def})\ \text{bel}_S((-\infty,x])\ . \tag{10}$$

In turn, define the *lower and upper expectations* of S, respectively, as (integration by parts forces the reversal of upper and lower forms)

$$E_*(S) =(\text{def}) \int_{x=-\infty}^{+\infty} x\ dF^*_S(x)\ , \qquad E^*(S) =(\text{def}) \int_{x=-\infty}^{+\infty} x\ dF_{*S}(x)\ . \tag{11}$$

Then, Dempster's main results, in light of the above definitions, again using functional image notation as in eq.(8), now also applied to the inequalities, become

$$\text{bel}_S(A) \le Q_S(A) \le \text{plaus}_S(A)\ ,\ \text{all } A \text{ in } \mathcal{B}\ , \tag{12}$$

$$E_*(S) \le E(S) \le E^*(S)\ . \tag{13}$$

In turn, Dubois & Prade, recognizing this, in effect, and restricting themselves to only the uniform random level set $S = S_{f,U}$, for given fuzzy interval f of D, noting the interesting relations, for all x in [0,1]

$$F^*_{S_{f,U}}(x) = \sup(f^{-1}[0,x]\ ),\ \ F_{*S_{f,U}}(x) = \inf((1\text{-}f)^{-1}(x,1]), \tag{14}$$

defined and obtained that

$$E(f) =(\text{def})\ E(S_{f,U}) = [E_*(S_{f,U})\ ,\ E^*(S_{f,U})\ ]\ . \tag{15}$$

Dubois & Prade also showed among other properties that the definition obeyed linearity with respect to addition of fuzzy intervals.

On the other hand, Chanas and Nowakowski (and equivalently, Kaufmann & Gupta) obtained the single value, via a two stage randomization, where $U_1$ and $U_2$ are statistically independent random variables each uniformly distributed over [0,1]

$$\begin{aligned} GE(f) &=(\text{def})\ E((1\text{-}U_1)\inf(f^{-1}[U_2,1]) + U_1\sup(f^{-1}[U_2,1])) \\ &= \int_{t=0}^{1} (\ \inf(f^{-1}[t,1]) + \sup(f^{-1}[t,1])\ )\ dt \\ &= (1/2)(E_*(S_{f,U}) + E^*(S_{f,U})\ )\ , \end{aligned} \tag{16}$$

the last equation being pointed out by Heilpern [18].

Most importantly, Heilpern also showed ([18], Theorem 1) that the bounds $E_*(S_{f,U})$ and $E^*(S_{f,U})$ remain the same when $S_{f,U}$ is replaced by any other random subset S of D which is closed interval-valued and one-point coverage-equivalent to f. (In Hailpern's work, D need not be finite.) As Dubois & Prade, he also demonstrates full linearity of E(f) with respect to fuzzy interval addition and scalar multiplication, as well.

González, motivated by the ranking of fuzzy sets, extended Dubois & Prade's as well as Hailperin's approach - called in the literature the FM (fuzzy mean) approach. (See also the previously-mentioned paper by Zhao and Govind [3] for comments on the FM method along with the COA and MOM ones.) Specifically, given some choice of additive measure M over [0,1] and t in [0,1], modifying his notation,

$$AV(f;\, t,M) =(\text{def}) \int_{s=0}^{1} (t\cdot \inf(f^{-1}[s,1] + (1-t)\cdot \sup(f^{-1}[s,1]))dM(s), \qquad (17a)$$

$$E(f;M) =(\text{def})\{AV(f;t,M):0\leq t\leq 1\}=[AV(f;1,M)\,,\, AV(f;0,M)] \qquad (17b)$$

González showed for M=identity (i.e., lebesgue) measure, the Dubois & Prade FM definition was obtained. He also showed connections with a number of other approaches and obtained various properties for this class of measures of central tendency. Mention should also be made of the recent work of Filev and Yager [22] toward defuzzification via the introduction of the concept generalized level set defuzzification (LSD) for discrete fuzzy sets, whereby a parameter $\alpha$ is prechosen and a resulting weighted average of the average value of each level set of f is obtained, where the weight of each level set is proportional to the number of elements in that level set multiplied by some suitable power of $\alpha$. Filev and Yager show this approach encompasses both the MON and COA approaches modified to level set considerations.

## 4. New Approach to Problem Using One-Point Coverage Relation Between Fuzzy Sets and Random Sets

With all of the above background established, it is clear that the problem of obtaining measures of central tendency of a given fuzzy set in terms of one-point coverage representations is a rather difficult one, except when restricted to fuzzy intervals. Theoretically, one could simply define for a given fuzzy subset f of finite set D - not necessarily a fuzzy interval - one natural measure as mentioned earlier (Goodman[5])

$$E_1(f) =(\text{def}) \sum_{A\subseteq D} \text{centroid}(A)\cdot P(S=A) , \tag{18}$$

or as Dubois & Prade and Hailpern did for fuzzy intervals,

$$E_2(f) =(\text{def})\ E(S_f) , \tag{19}$$

or the larger interval, in general

$$E_3(f) =(\text{def})\ [E_*(S_f) , E^*(S_f)] . \tag{20}$$

Finally, one could also consider mode-like or maximum likelihood-like estimations corresponding to the MOM approach, but based on random set representations such as

$$E_4(f) =(\text{def})\ A \text{ for which } \max \{P(S_f = A): A\subseteq D\} \text{ holds.} \tag{21}$$

But, which one-point coverage equivalent random set $S_f$ to pick ? It is not clear that there is only one such natural choice - as e.g., shown by Heilpern (mentioned earlier).

A different tack to the above problem is taken here: We consider a *natural class* of one-point coverage-equivalent random sets to f (still not necessarily a fuzzy interval), compute if feasible the above measures of central tendency, and then analyze the resulting values. The class in question arises from the following problem:

Find all copula, cocopula pairs ($g$ , $h$ ) such that arbitrary finite combinations of conjunctions and disjunctions of fuzzy sets correspond homomorphically to the probability of such combinations of the one-point coverage relations of at least some random sets, each of which is one-point coverage equivalent to its corresponding fuzzy set:

$$P(\text{ comb}(\&,\text{or})(x_{ij} \text{ in } S_{f_{ij}}; i \text{ in } I, j \text{ in } J)) = \text{comb}(g,h)(f_{ij}(x_{ij}); i \text{ in } I, j \text{ in } J), \tag{22}$$

for all finite index sets I,J, all fuzzy sets $f_{ij}:D_{ij} \to [0,1]$, all $x_{ij}$ in $D_{ij}$. $(S_{f_{ij}})_{i \text{ in } I, j \text{ in } J}$ is some joint collection of one-point coverage-equivalent random subsets of the $D_{ij}$ to the $f_{ij}$, all i in I, j in J. In [0] it is shown (see, especially Theorem 4.1 and Corollary 4.1), assuming the $D_{ij}$ are all finite and making additional mild assumptions, that the solution class $(g,h)$ to eq.(22) is precisely the disjunction of the following relations:

$$\begin{aligned}
&\text{Case (i): } (g,h) = (\min,\max) , \\
&\text{Case (ii): } (g,h) = (\text{prod},\text{probsum}) , \\
&\text{Case (iii): } (g,h) \text{ in } \{\text{ordsum}(\text{prod},\text{probsum}): \text{all ordsums}\}.
\end{aligned} \tag{23}$$

Here, min(imum), max(imum),and prod(uct) are self-explanatory and probsum is the deMorgan transform of prod, while ordsum means "ordinal sum". A typical ordsum in (iii) is described as, for any $\underline{x}$ in $[0,1]^n$, $n \geq 1$, and any choice of finite or countably infinite index set N of disjoint closed subintervals $[a_k,b_k]$ of [0,1],

$$g(\underline{x}) = L(a_k,b_k)^{-1}(\mathrm{prod}(L(a_k,b_k)(\underline{x}))) ,$$
$$h(\underline{x}) = L(a_k,b_k)^{-1}(\mathrm{probsum}(L(a_k,b_k)(\underline{x}))) , \qquad (24)$$

if there is a (unique) interval $[a_k,b_k]$ such that $\underline{x}$ in $[a_k,b_k]^n$ , for some k in N;

$$g(\underline{x}) = \min(\underline{x}) , \quad h(\underline{x}) = \max(\underline{x}) , \qquad (25)$$

if $\underline{x}$ not in any $[a_k,b_k]$, all k in N , where multivariable affine transform

$$L(a_k,b_k)(\underline{x}) = ((x_1-a_1)/(b_1-a_1),\ldots,(x_n-a_n)/(b_n-a_n)) \qquad (26)$$

with single variable inverse

$$L(a_k,b_k)^{-1}(y) = a_k + (b_k-a_k)\cdot y, \text{ all y in } [0,1]. \qquad (27)$$

Note also that Case (ii) can be considered as a limiting form of Case (iii) with N = {1} and $[a_1,b_1] = [0,1]$. (See, e.g. Dall'Aglio et al. [23] for further properties of, and background for, copulas, cocopulas, t-norms, t-conorms, ordinal sums, etc.) In turn, Cases (i)-(iii) can be translated into the necessary joint structure that the corresponding one-point coverage-equivalent random sets $S_{f_{ij}}$ must take, as zero-one stochastic processes determined up to joint distributions with respect to the different $x_{ij}$ in $D_{ij}$ (again, see [0]). By also assuming the random sets in question are completely controlled by the copulas in Cases (i)-(iii), we obtain finally (via [0], Appendix, eq.(iii)) the explicit marginal probability distribution of any such possible random set $S_f$ as:

$$P(S_f = A) = h((1-f(x))_{x \text{ in } A}, g((1-f(x))_{x \text{ in } D-A})) - h((1-f(x))_{x \text{ in } A}), \qquad (28)$$

for all $A \subseteq D$. In turn, eq.(28) becomes for each pair ($g,h$ ) in the above cases:

*Case(i)*: $P(S_f = A) = P(S_{f,U} = A)$ , for all $A \subseteq D$.

Hence,

$$P(S_f = f^{-1}[\alpha_i,1]) = \alpha_i - \alpha_{i+1} , \; i=1,..,r, \qquad (29)$$

where

$$\mathrm{range}(f) =(\mathrm{def}) \; \{\alpha_i: i=1,..,r\}; \; 1 \geq \alpha_1 >..> \alpha_r \geq 0 = \alpha_{r+1} . \qquad (30)$$

*Case(ii)*: Treat this -apropos to the above comment - as a special case of Case (iii)

*Case(iii), eq.(25) holding*: Treat this as in Case (i).

Case(iii), *eq.(24) holding* for some $[a_k, b_k]$, k in N. Hence,

$$a_k \le 1\text{-}f(x) \le b_k \text{ , all x in D ,}$$

i.e.,

$$1-b_k \le f(D) \le 1\text{-} a_k \text{ .} \quad (31)$$

Simplifying eq.(28) yields for any $A \subseteq D$,

$$P(S_f = A) = c(f;a_k,b_k)\cdot q(A;f;b_k,a_k) \text{ ,}$$

where

$$q(A;f;b_k,a_k) = (\text{def}) \text{ prod}( (b_k\text{-}(1\text{-}f(x))) / (1\text{-}f(x)\text{-}a_k); \text{x in A}) ), \quad (32)$$

and the constant (independent of A)

$$c(f;a_k,b_k) = (\text{def}) (b_k\text{-}a_k)\cdot\text{prod}( (1\text{-}f(x)\text{-}a_k) / (b_k\text{-}a_k); \text{x in D}). \quad (33)$$

If we consider, e.g. the maximum criterion $E_4(f)$ as in eq.(21), it is clear that reasonable closed-form expressions can be obtained for all of the above cases:

*Case(i)*: Obtain that (or those) $A = f^{-1}[\alpha_i,1]$ corresponding to that (or those) i for which max $(\alpha_i - \alpha_{i+1}$ ; i=1,..,r } occurs.

*Case (iii), eq.(24) holding*:

*Subcase(I)*: $q(A;f;b_k,a_k) \ge 1$ , for some x in D, which is equivalent to,

$$1\text{-} ((a_k+b_k)/2) \le f(x), \text{ for some x in D} \quad (34)$$

Then, in light of (31),choose

$$A = \{\text{x in D}: q(A;f;b_k,a_k) \ge 1\} = f^{-1}[1\text{-} ((a_k+b_k)/2) \text{ , } 1\text{-}a_k] \text{ .} \quad (35)$$

*Subcase(II)*: $q(A;f;b_k,a_k) < 1$ , for all x in D, which is equivalent to,

$$1\text{-} ((a_k+b_k)/2) > f(x), \text{ for all x in D,} \quad (36)$$

which by (31) becomes

$$1\text{-}b_k \le f(D) < 1\text{-} ((a_k+b_k)/2) \text{ .} \quad (37)$$

In turn, this implies (since each $q(\{x\};f;b_k,a_k)$ is a monotone increasing function of f(x))

that we should choose

$$A = \{x\}, \text{ for that x corresponding to } \max(f(x); x \text{ in } D). \qquad (38)$$

Further analysis can be carried out in the above vein and future work will detail this. In summary, the most promising results, from a feasible computational viewpoint, appear to be derived via $E_4(f)$, as opposed to the other criteria. Finally, some progress has been made toward relating a number of ad hoc approaches to measures of central tendency, such as the COA type, and this approach.

# 5. Conclusions

This paper, first, has attempted to present a broad view of previous efforts in treating measures of central tendency of fuzzy sets. Noting that the natural identification of any given fuzzy set with the *full* class of all one-point coverage-equivalent random subsets of the fuzzy set membership function's domain, a good deal of this research has been devoted to obtaining measures of central tendency (such as expectation) of certain types of these random sets. In particular, the nested random set which is one-point coverage-equivalent to a fuzzy set has been investigated rather thoroughly, such as by Dubois & Prade. But, in general, the class of all other one-point coverage equivalent random sets (to a given fuzzy set) is quite large and results, therefore, in random sets which can have a wide variation of measures of central tendency, and thus may not produce a satisfactory "tight" set of representative values for central tendency. By restricting fuzzy sets to be fuzzy numbers or fuzzy intervals, stronger results can be obtained for the entire class of one-point coverage-equivalent random sets - such as that obtained by Heilpern. On the other hand, the approach taken in the last part of this paper does *not* restrict the form of fuzzy sets considered -- except to assume the domain of the given fuzzy set to be finite -- but rather, considers a natural subclass of the entire class of one-point coverage-equivalent random sets. To this end, it appears that one can derive relatively simple forms for the probability functions of this class. In turn, these probability functions appear to have relatively simple forms, and thus have the potential for yielding relatively feasible forms for their corresponding measures of central tendency.

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